Variational Principles for the Elastodynamics of Rotating Planets

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❖Main results

❖ We have derived expressions describing the gravitational interaction of two arbitrarily shaped, rotating bodies

❖ Self-gravity is included completely and finite deformations are allowed

❖ Extension to many bodies should be straightforward

❖ Fluid-solid boundaries are analysed

❖Motivation

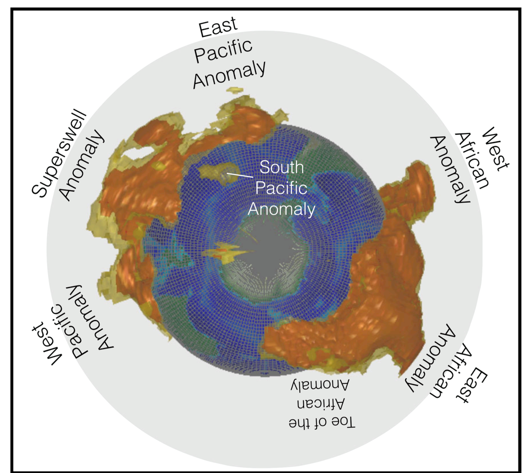
❖ Long period free-oscillations are a useful probe of density structure — self-gravity and rotation are crucial!

❖ Second- and higher-order effects of self-gravity might be important

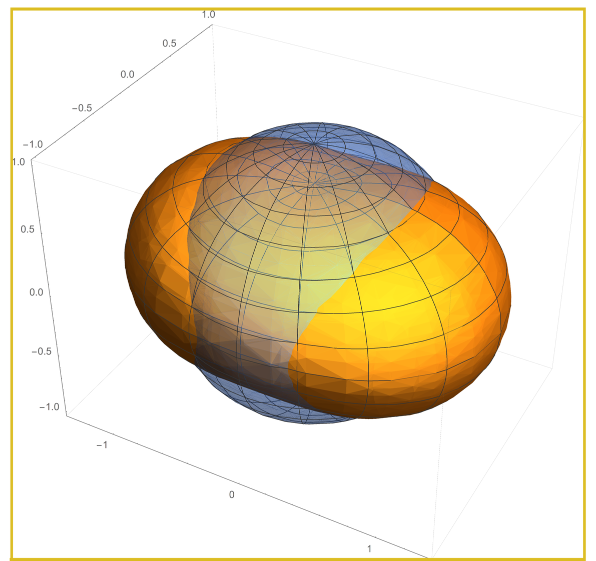
❖ Avoids introducing approximations early on

❖ Motion decomposed so as to facilitate both computation and comprehension

❖ Interesting physics 😊



Tomographic image of LLSVPs. At extremely long periods free-oscillations are sensitive to long-wavelength variations in density. Now that the Earth's spherically symmetric structure is fairly well-known we hope that normal modes will be useful in probing lateral density variations. From (Cottaar & Lekic, 2016).



Equilibrium figure of a solid planet of uniform density in hydrostatic equilibrium tidally-locked with an identical body. The planets assume a counterintuitive shape resembling a peanut! The perturbation parameter has been made unrealistically large so as to make the shape clear.

❖An interesting calculation

❖ Take the case of two solid bodies (see right) and enforce tidal-locking and hydrostatic equilibrium

❖ Assume the undeformed bodies as identical spheres and calculate their equilibrium topography perturbatively.

❖ To first order the topography is expressed in terms of the (unknown) gravitational and (known) centrifugal/tidal potentials as

$$h^{(1)} = -\frac{1}{\partial_r \phi^{(0)}|_{r=a}} \left(\phi^{(1)}(a) + \psi^{(1)}(a) \right)$$

❖ Finally solve "Clairaut's equation" for the perturbed gravitational potential:

$$\nabla^2 \phi^{(1)} - 4\pi G \frac{\partial_r \rho^{(0)}}{\partial_r \phi^{(0)}} \left(\phi^{(1)} + \psi^{(1)} \right) = 0\,,$$

$$\left[\phi^{(1)} \right]_-^+ = \left[\hat{\mathbf{r}} \cdot \nabla \phi^{(1)} - 4\pi G \frac{\rho^{(0)}}{\partial_r \phi^{(0)}} \left(\phi^{(1)} + \psi^{(1)} \right) \right]_-^+ = 0$$

❖Relevance of rotation and self-gravity

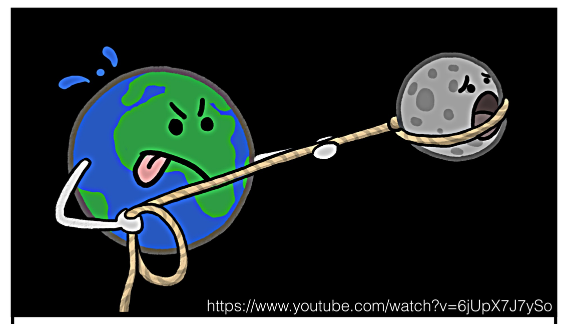
❖ Scaling behaviour of the various parameters:

$$L \sim \bar{v} T \underbrace{\left(1 + 3\,\Omega\, T + \Omega^2\, T^2\right)^{-\frac{1}{2}}}_{\text{rotation}} \underbrace{\left(1 + \frac{4\pi G \bar{\rho}}{\bar{v}^2/L^2}\right)^{\frac{1}{2}}}_{\text{self-gravity}}$$

❖ For the whole Earth ($L \sim R_\oplus$):

$$\frac{4\pi G\bar{\rho}}{\bar{v}^2/L^2} \sim \frac{\text{gravitational forcing}}{\text{elastic forcing}} \sim 3$$
$$\Omega\, T \sim \frac{\text{Coriolis forces}}{\text{inertial forces}} \sim \frac{T}{4\text{hrs}}$$

❖ Deformations occurring on typical timescale of $\lesssim \sqrt{4\pi G \bar{\rho}} \sim 430\, s$ shouldn't be affected by self-gravity



"The tethered moon". The moon's orbital history is still hotly debated, *cf. e.g.* (Zahnle *et al.*, 2015). We hope that our work might be able to shed some light on this problem as well as others, perhaps in the field of exoplanets, where objects cannot be approximated as purely solid/liquid spheres, ellipses, discs etc. *cf. e.g.* (Lock *et al.*, 2017). A question of particular interest to the present authors concerns the effect of the moon on the Earth's long period free-oscillations, which would in principle involve only a small extension of this work.

❖Decomposition of the motion

❖ Standard formulation of continuum mechanics with the motion defined by the function $\boldsymbol{\varphi}(\mathbf{x},t)$ from a reference body to physical space

❖ Decompose into CoM motion & relative rotation **with superimposed elastic deformation**:

$$\boldsymbol{\varphi}(\mathbf{x},t) = \boldsymbol{\varphi}_c(t) + \mathbf{R}(t) \cdot \boldsymbol{\varphi}_r(\mathbf{x},t)$$
$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_c(t) + \mathbf{R}(t) \cdot [\mathbf{v}_r(t) + \boldsymbol{\Omega}(t) \times \boldsymbol{\varphi}_r(\mathbf{x},t)]$$

❖ No change to the information! Just expressing it in a different **and ultimately more informative** manner. See below.

❖Single solid, rotating, self-gravitating body

❖ Action:

$$\mathcal{S} = \int d^4x \Big\{ \frac{1}{2}\rho \mathbf{v}_c^2 + \frac{1}{2}\rho \mathbf{v}_r^2 + \frac{1}{2}\rho \|\boldsymbol{\Omega} \times \boldsymbol{\varphi}_r\|^2 - W(\mathbf{x}, \mathbf{F}_r)$$
$$+ \frac{1}{2} G\rho \int_M \frac{\rho(\mathbf{x}')}{\|\boldsymbol{\varphi}_r(\mathbf{x},t) - \boldsymbol{\varphi}_r(\mathbf{x}',t)\|} \mathrm{d}^3\mathbf{x}'$$
$$+ \langle \boldsymbol{\alpha}, \rho\boldsymbol{\varphi}_r \times \mathbf{v}_r \rangle + \langle \boldsymbol{\beta}, \rho \boldsymbol{\varphi}_r \rangle \Big\}$$

❖ Equations of motion:

$$\rho \frac{\partial \mathbf{v}_r}{\partial t} - \nabla \cdot \mathbf{T}_r - \rho \boldsymbol{\gamma}_r + \rho \left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\varphi}_r) + 2\boldsymbol{\Omega} \times \mathbf{v}_r + \dot{\boldsymbol{\Omega}} \times \boldsymbol{\varphi}_r \right) = 0$$
$$\partial_t \left(\mathbb{I}_r \cdot \boldsymbol{\Omega} \right) + \boldsymbol{\Omega} \times \left(\mathbb{I}_r \cdot \boldsymbol{\Omega} \right) = 0\,,$$
$$\mathbf{v}_c = \text{const.}\,,$$

$$\text{Constraints: } \int \rho \boldsymbol{\varphi}_r \mathrm{d}^3\mathbf{x} = \int \rho \boldsymbol{\varphi}_r \times \mathbf{v}_r \mathrm{d}^3\mathbf{x} = 0\,,$$
$$\text{Boundary conditions: } \mathbf{T}_r \cdot \hat{\mathbf{n}} = 0\,,$$

$$\boldsymbol{\gamma}_r \equiv -G \int_M \rho' \frac{\boldsymbol{\varphi}_r - \boldsymbol{\varphi}_r'}{\|\boldsymbol{\varphi}_r - \boldsymbol{\varphi}_r'\|^3} \mathrm{d}^3\mathbf{x}'$$

❖ Rigid-body motion coupled to "net momentum-free" elastodynamics

❖ More complicated to look at but clearer what is happening physically. *Cf.*

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot \mathbf{T} - \rho \boldsymbol{\gamma} = 0$$

❖Two solid, rotating, self-gravitating bodies

❖ Make the problem look like the particle two-body problem (and thus Kepler's problem!) by writing $\boldsymbol{\Psi} = \boldsymbol{\varphi}_1^c - \boldsymbol{\varphi}_2^c$

❖ Couple two single-body actions by adding them together and then adding the mutual gravitational potential:

$$G \int \mathrm{d}^3\mathbf{x}_1 \int \mathrm{d}^3\mathbf{x}_2 \frac{\rho_1 \rho_2}{\|\boldsymbol{\Psi} + \mathbf{R}_1 \cdot \boldsymbol{\varphi}_1^r - \mathbf{R}_2 \cdot \boldsymbol{\varphi}_2^r\|}$$

❖ Equations of motion:

$$\mu \ddot{\boldsymbol{\Psi}} + \int \mathrm{d}^3\mathbf{x}_1 \rho_1 \boldsymbol{\Gamma}_2 = 0\,,$$
$$\rho_i \ddot{\boldsymbol{\varphi}}_i^r - \nabla_i \cdot \mathbf{T}_i^r - \rho_i \boldsymbol{\gamma}_i^r + \rho_i \left[\boldsymbol{\Omega}_i \times (\boldsymbol{\Omega}_i \times \boldsymbol{\varphi}_i^r) + 2\boldsymbol{\Omega}_i \times \dot{\boldsymbol{\varphi}}_i^r + \dot{\boldsymbol{\Omega}}_i \times \boldsymbol{\varphi}_i^r \right] \pm \rho_i \mathbf{R}_i^{\mathbb{T}} \cdot \left[\boldsymbol{\Gamma}_j - \frac{1}{\mathcal{M}_i} \int \mathrm{d}^3\mathbf{x}_i \rho_i \boldsymbol{\Gamma}_j \right] = 0\,,$$
$$(\partial_t + \boldsymbol{\Omega}_i \times) \left(\mathbb{I}_i^r \cdot \boldsymbol{\Omega}_i \right) \pm \int \mathrm{d}^3\mathbf{x}_i \rho_i \boldsymbol{\varphi}_i^r \times \left(\mathbf{R}_i^{\mathbb{T}} \cdot \boldsymbol{\Gamma}_j \right) = 0$$

❖ Orbital motion, rigid-body motion and "simplified" elastodynamics all coupled together (i=1,2) — internal deformation influencing orbital motion explicitly

❖ For spherical bodies $\int \mathrm{d}^3\mathbf{x}_1 \rho_1 \boldsymbol{\Gamma}_2 = \frac{G\mathcal{M}\mu}{\Psi^2} \hat{\boldsymbol{\Psi}}$ … *cf.* Kepler's problem!

❖ Tidal forcing drops out of the analysis naturally

❖Single solid, rotating, self-gravitating body with a fluid core

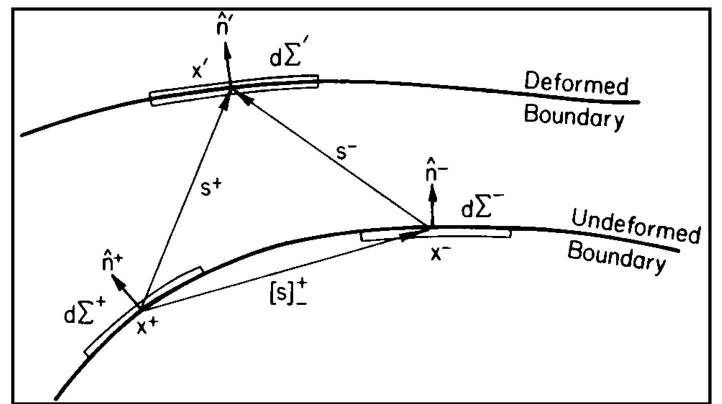
❖ In the two-body system described above take body 1 to reside inside body 2 and add to the action a Lagrange-multiplier term which enforces tangential slip along the boundary between the bodies:

$$\int_\Sigma \mathrm{d}S\, \Pi\, \sigma \circ \boldsymbol{\varphi}_1^{-1} \circ \boldsymbol{\varphi}_2$$

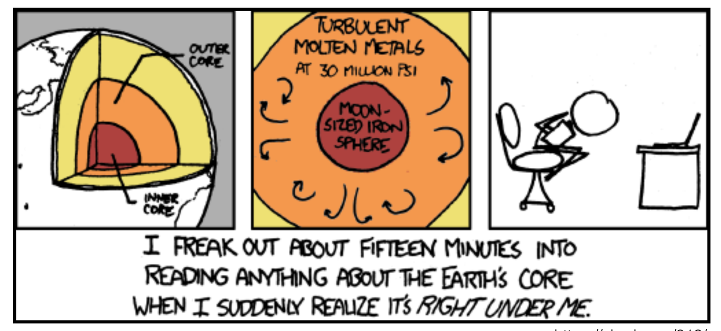
❖ Equations of motion are just as above, but with forcing on the RHS due to net forces and torques on the internal boundary:

$$\mu \ddot{\boldsymbol{\Psi}} + \ldots = \mathbf{R}_1 \cdot \int_\Sigma \mathrm{d}S\, \mathbf{t}_1^r$$
$$\rho_i \ddot{\boldsymbol{\varphi}}_i^r + \ldots = \mp \frac{1}{\mathcal{M}_i} \int_\Sigma \mathrm{d}S\, \mathbf{t}_i^r$$
$$\partial_t \left(\mathbb{I}_i^r \cdot \boldsymbol{\Omega}_i \right) + \ldots = \pm \int_\Sigma \mathrm{d}S\, \boldsymbol{\varphi}_i^r \times \mathbf{t}_i^r$$

❖ The "orbital" component of this system should be very small. It represents the motion of the respective CoMs of the "core" and "mantle" about one another. Scope for approximation…



Fluid-solid boundaries in continuum mechanics. Tangential slip occurs at the boundary so the points either side of the boundary can be far apart in reference space; this has caused difficulties historically. Here we use a novel method to model tangential slip from (Al-Attar *et al.*, unpublished). Figure from (Woodhouse & Dahlen, 1978).



https://xkcd.com/913/

❖Numerical solution — for the future

❖ Direct solution of the equations as written is hard due to time-dependence under the integrals

❖ I am currently working on an analytical expansion of the net force term, although this is unlikely to be efficient to compute

❖ More realistically, I am writing a code to calculate the gravitational potential of an arbitrarily shaped body by solving the Poisson equation using a Dirichlet-to-Neumann map

❖ Runge-Kutta solvers will advance the orbital and rigid-body equations. These will be coupled to spectral-element codes to solve the equations of elastodynamics on each body at each time-step

❖ The problem involves a perturbed Kepler problem so the method of osculating orbits should be useful

❖ For bodies that deviate only slightly from spherical symmetry two-timescale analysis could be used to isolate secular orbital changes

❖References

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❖Thanks

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