Variational Principles for the Elastodynamics of Rotating Planets

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❖Main results

- ❖We have derived expressions describing the gravitational interaction of two arbitrarily shaped, rotating bodies
- ❖Self-gravity is included completely and finite deformations are allowed
- ❖Extension to many bodies should be straightforward
- ❖Fluid-solid boundaries are analysed

❖Motivation

- ❖Long period free-oscillations are a useful probe of density structure — self-gravity and rotation are crucial!
- ❖Second- and higher-order effects of self-gravity might be important
- ❖Avoids introducing approximations early on
- ❖Motion decomposed so as to facilitate both computation and comprehension
- ❖Interesting physics 😊

❖An interesting calculation

- ❖Enforce tidal-locking and hydrostatic equilibrium on two interacting solid bodies
- ❖Assume the undeformed bodies as identical spheres and calculate their equilibrium topography perturbatively.
- ❖To first order the topography is expressed in terms of the (unknown) gravitational and (known) centrifugal/tidal potentials as

$$h^{(1)} = -\frac{1}{\partial_r \phi^{(0)}|_{r=a}}\left(\phi^{(1)}(a) + \psi^{(1)}(a)\right)$$

- ❖Finally solve "Clairaut's equation" for the perturbed gravitational potential:

$$\nabla^2 \phi^{(1)} - 4\pi G \frac{\partial_r \rho^{(0)}}{\partial_r \phi^{(0)}}\left(\phi^{(1)} + \psi^{(1)}\right) = 0\,,$$

$$\left[\phi^{(1)}\right]_-^+ = \left[\hat{\mathbf{r}} \cdot \nabla \phi^{(1)} - 4\pi G \frac{\rho^{(0)}}{\partial_r \phi^{(0)}}\left(\phi^{(1)} + \psi^{(1)}\right)\right]_-^+ = 0$$

❖Relevance of rotation and self-gravity

- ❖Scaling behaviour of the various parameters:

$$L \sim \bar{v}T \underbrace{\left(1 + 3\Omega T + \Omega^2 T^2\right)^{-\frac{1}{2}}}_{\text{rotation}} \underbrace{\left(1 + \frac{4\pi G \bar{\rho}}{\bar{v}^2/L^2}\right)^{\frac{1}{2}}}_{\text{self-gravity}}$$

- ❖For the whole Earth ($L \sim R_\oplus$):

$$\frac{4\pi G\bar{\rho}}{\bar{v}^2/L^2} \sim \frac{\text{gravitational forcing}}{\text{elastic forcing}} \sim 3$$

$$\Omega T \sim \frac{\text{Coriolis forces}}{\text{inertial forces}} \sim \frac{T}{4\text{hrs}}$$

- ❖Deformations occurring on typical timescale of ≲430s shouldn't be affected by self-gravity

❖Decomposition of the motion

- ❖Decompose into CoM motion & relative rotation **with superimposed elastic deformation**:

$$\boldsymbol{\varphi}(\mathbf{x},t) = \boldsymbol{\varphi}_c(t) + \mathbf{R}(t)\cdot\boldsymbol{\varphi}_r(\mathbf{x},t)$$
$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_c(t) + \mathbf{R}(t)\cdot\left[\mathbf{v}_r(t) + \boldsymbol{\Omega}(t)\times\boldsymbol{\varphi}_r(\mathbf{x},t)\right]$$

❖Single solid, rotating, self-gravitating body

- ❖Action:

$$\mathcal{S} = \int d^4x \left\{ \frac{1}{2}\rho \mathbf{v}_c^2 + \frac{1}{2}\rho\mathbf{v}_r^2 + \frac{1}{2}\rho\|\boldsymbol{\Omega}\times\boldsymbol{\varphi}_r\|^2 - W(\mathbf{x},\mathbf{F}_r) + \frac{1}{2}G\rho\int_M \frac{\rho(\mathbf{x}')}{\|\boldsymbol{\varphi}_r(\mathbf{x},t) - \boldsymbol{\varphi}_r(\mathbf{x}',t)\|}\mathrm{d}^3\mathbf{x}' + \langle\boldsymbol{\alpha}, \rho\boldsymbol{\varphi}_r\times\mathbf{v}_r\rangle + \langle\boldsymbol{\beta}, \rho\boldsymbol{\varphi}_r\rangle \right\}$$

- ❖Equations of motion:

$$\rho\frac{\partial \mathbf{v}_r}{\partial t} - \nabla\cdot\mathbf{T}_r - \rho\gamma_r + \rho\left(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{\varphi}_r) + 2\boldsymbol{\Omega}\times\mathbf{v}_r + \dot{\boldsymbol{\Omega}}\times\boldsymbol{\varphi}_r\right) = 0$$
$$\partial_t\left(\mathbb{I}_r\cdot\boldsymbol{\Omega}\right) + \boldsymbol{\Omega}\times\left(\mathbb{I}_r\cdot\boldsymbol{\Omega}\right) = 0\,,$$
$$\mathbf{v}_c = \text{const.}\,,$$

$$\text{Constraints: } \int \rho\boldsymbol{\varphi}_r \mathrm{d}^3\mathbf{x} = \int \rho\boldsymbol{\varphi}_r\times\mathbf{v}_r\mathrm{d}^3\mathbf{x} = 0\,,$$
$$\text{Boundary conditions: } \mathbf{T}_r\cdot\hat{\mathbf{n}} = 0\,,$$

$$\gamma_r \equiv -G\int_M \rho' \frac{\boldsymbol{\varphi}_r - \boldsymbol{\varphi}_r'}{\|\boldsymbol{\varphi}_r - \boldsymbol{\varphi}_r'\|^3}\mathrm{d}^3\mathbf{x}'$$

- ❖Rigid-body motion coupled to "net momentum-free" elastodynamics
- ❖More complicated to look at but clearer what is happening physically.

❖Two solid, rotating, self-gravitating bodies

- ❖Make the problem look like the particle two-body problem using CoM coordinates
- ❖Couple two single-body actions using the mutual gravitational potential:

$$G\int \mathrm{d}^3\mathbf{x}_1 \int \mathrm{d}^3\mathbf{x}_2 \frac{\rho_1\rho_2}{\|\boldsymbol{\Psi} + \mathbf{R}_1\cdot\boldsymbol{\varphi}_1^r - \mathbf{R}_2\cdot\boldsymbol{\varphi}_2^r\|}$$

- ❖Equations of motion:

$$\mu\ddot{\boldsymbol{\Psi}} + \int \mathrm{d}^3\mathbf{x}_1\rho_1\boldsymbol{\Gamma}_2 = 0\,,$$
$$\rho_i\ddot{\boldsymbol{\varphi}}_i^r - \nabla_i\cdot\mathbf{T}_i^r - \rho_i\gamma_i^r + \rho_i\left[\boldsymbol{\Omega}_i\times(\boldsymbol{\Omega}_i\times\boldsymbol{\varphi}_i^r) + 2\boldsymbol{\Omega}_i\times\dot{\boldsymbol{\varphi}}_i^r + \dot{\boldsymbol{\Omega}}_i\times\boldsymbol{\varphi}_i^r\right] \pm \rho_i\mathbf{R}_i^{\mathbb{T}}\cdot\left[\boldsymbol{\Gamma}_j - \frac{1}{\mathcal{M}_i}\int \mathrm{d}^3\mathbf{x}_i\rho_i\boldsymbol{\Gamma}_j\right] = 0\,,$$
$$(\partial_t + \boldsymbol{\Omega}_i\times)\left(\mathbb{I}_i^r\cdot\boldsymbol{\Omega}_i\right) \pm \int \mathrm{d}^3\mathbf{x}_i\rho_i\boldsymbol{\varphi}_i^r\times\left(\mathbf{R}_i^{\mathbb{T}}\cdot\boldsymbol{\Gamma}_j\right) = 0$$

- ❖Orbital motion, rigid-body motion and "simplified" elastodynamics all coupled together (i=1,2) — internal deformation influencing orbital motion explicitly
- ❖For spherical bodies $\int \mathrm{d}^3\mathbf{x}_1\rho_1\boldsymbol{\Gamma}_2 = \frac{G\mathcal{M}\mu}{\Psi^2}\hat{\boldsymbol{\Psi}} \ldots$
- ❖Tidal forcing drops out naturally

❖Single solid, rotating, self-gravitating body with a fluid core

- ❖Put body 1 inside body 2 and enforce tangential slip along the boundary between the bodies.
- ❖EoMs as above, but with forcing terms due to net forces and torques on the internal boundary.
- ❖The "orbital" component of this system should be very small: motion of respective CoMs of the "core" and "mantle" about one another.

❖Numerical solution — for the future

- ❖Direct solution difficult due to time-dependence under the integrals
- ❖Currently writing a code to calculate the gravitational potential of an arbitrarily shaped body by solving Poisson's equation using a Dirichlet-to-Neumann map
- ❖Use Runge-Kutta solvers to advance the orbital and rigid-body equations. These will be coupled to spectral-element codes to solve the equations of elastodynamics on each body at each time-step
- ❖Perturbed Kepler problem — use the method of osculating orbits?
- ❖Employ two-timescale analysis to isolate secular orbital changes